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Weighting function scheme and its application on multidimensional conservation equations

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Abstract—The weighting function scheme proposed previously has shown great success in solving physical problems without a conservative form such as the wave instability problems and the non-similarity boundary layer flow equations. However, in the previous formulation for the weighting function scheme, the grid is restricted to uniform step size and a modification must be made to force the scheme to obey an important numerical rule. In the present investigation, a new formulation is proposed to reformulate the weighting function scheme such that the constraint of uniform grid can be removed. In addition, the new formulation guarantees the weighting function scheme to satisfy the numerical rule without the need of any further assumption. When applied to conservation equations, the weighting function scheme is seen to become Patankar's exponential scheme for uniform thermal conductivity cases. For cases of variable thermal conductivity, however, the weighting function scheme has a performance superior to that of Patankar's exponential scheme. The weighting function scheme thus is expected to be good for use in solving a turbulent flow near a wall where the eddy viscosity possesses a sharp variation due to the existence of a viscous sublayer adjacent to the solid surface.

INTRODUCTION

THERE seems to be little doubt that the methods of classical mathematics do not offer a practical way for solving the complicated differential equations that arise in practical engineering problems and in nature. Fortunately, the development of numerical methods and large digital computers during the past decades has shown promise in solving almost any practical problem. In the earliest attempts to solve convective heat transfer problems, the central difference scheme was developed based on a Taylor series expansion to cast the differential equation into a system of algebraic equations. However, the central difference scheme might produce unrealistic results when the magnitude of the convection term is large. This is why all of the early studies based on the central difference scheme were restricted to low Reynolds numbers.

To remedy the numerical difficulty encountered, many numerical schemes such as the upwind difference scheme [1], the hybrid scheme [2], the exponential scheme [2–4], and the power-law scheme [4, 5] have been proposed. Although the upwind difference scheme does not produce unrealistic results, its accuracy is not satisfactory especially when the convection is not strong. In 1972, Spalding [2] obtained an analytical solution from a steady Burgers' equation having constant coefficients. Based on the analytical solution, Spalding [2] proposed a simple correlation known as the hybrid scheme. Subsequently, Raithby and Torrance [3] derived the exponential scheme out of Spalding's analytical solution. To account for the effect of variable thermal conductivity, Patankar [4] employed the harmonic mean procedure [6] to modify the exponential scheme for heat convection problems with variable thermal conductivity. Thus, Patankar's exponential scheme reduces to the harmonic mean scheme [6] for heat conduction problems. To conserve CPU time, Patankar [4, 5] presented an approximation known as the power-law to his exponential scheme.

It should be noted here that Patankar's exponential scheme [4] is established on the basis of a control volume approach, while an artificial thermal conductivity is employed instead of the real one at the surface of a control volume. Thus, the accuracy of Patankar's exponential scheme is of a questionable nature. Note also that Patankar's exponential scheme does not obey Rule 4 stated in Chapter 3 of his book [4] even when the thermal conductivity assumes a constant value. This might arise from the fact that, in a control volume approach, the heat flux at each point inside the control volume does not necessarily satisfy the conservation law, although it does between the control surfaces. To circumvent this difficulty, Patankar introduced the law of mass conservation into his exponential scheme to force his scheme to obey Rule 4 (see Section 5.3-1 of ref. [4]). As a result, the application of Patankar's exponential scheme is restricted to equations having conservative form. Unfortunately, many differential equations arising in physical problems such as the Blasius equation in boundary layer flow, the Orr–Sommerfeld equation in wave instability problems and the Poisson grid generation

NOMENCLATURE

a	coefficient of the first derivative term in equation (1) or a parameter for the variation in Γ
b	parameter for the variation of Γ in the y-coordinate
F	mass flow rate
$f_a(x)$	function defined by equations (28)
p	parameter of the p-method, see equations (20)
S_p, S_c	coefficients in the source term $(S_p\theta + S_c)$
t	time
$w_f(Z)$	weighting function, $Z/(1-e^{-Z})$
x, y, z	physical coordinates
Z	parameter of the weighting function
Z_x, Z_y	defined in equations (19).

Greek symbols

Γ	thermal conductivity
Δ	finite difference quantity
θ	temperature
ξ	$x - x_i$
ξ^*	$x - x_{i-1}$.

Subscripts

0	quantity at the previous time $t - \Delta t$
i, j, k	quantity at (x_i, y_j, z_k).

equations for generating a non-orthogonal grid system do not possess a conservative form. Under such a situation, a non-conservative method called the weighting function scheme was developed on a uniform grid system [7]. Unlike Patankar's exponential scheme, the weighting function scheme is derived mathematically rather than based on a control volume approach. Therefore, the weighting function scheme has a more flexible use than Patankar's exponential scheme does in fluid flow and heat transfer applications. In fact, the weighting function scheme has been applied satisfactorily to many problems without conservative form such as the wave instability problems [7–11] and the non-similarity boundary layer flow equations [12, 13].

Note that in the previous weighting function scheme formulation [7], a modification must be made to force the scheme to satisfy the above-mentioned Rule 4. In addition, the application of the weighting function scheme is restricted to a uniform grid. The purpose of the present investigation is to propose a new formulation based on a different point of view to reformulate the weighting function scheme. The new formulation is seen to successfully remove the constraint of a uniform grid on the weighting function scheme formulated previously [7]. In addition, Rule 4 can be satisfied automatically without the need of any further assumption. Therefore, when applied to multidimensional energy (or momentum) conservation equations, the weighting function scheme does not pose any constraint such as 'conservation of mass flow' required in the use of Patankar's exponential scheme. The performance of the weighting function scheme will be compared with that of Patankar's exponential scheme through a few general examples with an emphasis on arbitrarily strong variation in the thermal conductivity. As has been pointed out by Roache [14], the use of a non-conservative form for the variable thermal conductivity can produce more accurate results than the conservative form. The weighting function scheme thus is expected to have a superior performance, because the use of Patankar's exponential scheme is restricted to the conservative form.

NEW FORMULATION FOR THE WEIGHTING FUNCTION SCHEME

Consider a homogeneous second-order ordinary differential equation of the form

$$\theta'' + a\theta' = 0 \quad \text{for} \quad x_1 \leqslant x \leqslant x_m \tag{1}$$

where the primes denote derivatives with respect to x and the coefficient a is a given function of x. Let the domain be divided into $(m-1)$ intervals and let the following simple notation be used:

$$\begin{aligned} \theta_i &= \theta(x_i) && \text{for} \quad i = 1, 2, \ldots, m \\ \Delta x_i &= x_{i+1} - x_i && \text{for} \quad i = 1, 2, \ldots, m-1 \\ x_{i+1/2} &= (x_i + x_{i+1})/2 && \text{for} \quad i = 1, 2, \ldots, m-1 \end{aligned} \tag{2}$$

where x_i, $i = 1, 2, \ldots, m$ are the successive m points in the domain. If the step size Δx_i is sufficiently small, the known coefficient $a(x)$ in the interval $[x_i, x_{i+1}]$ can be approximated with a step function at $x_{i+1/2}$ as

$$a(x) = a_{i+1/2} = a(x_{i+1/2}) \quad \text{for} \quad x_i \leqslant x \leqslant x_{i+1}. \tag{3}$$

Figure 1 shows a schematic profile of the function $a(x)$ as well as its step function approximation.

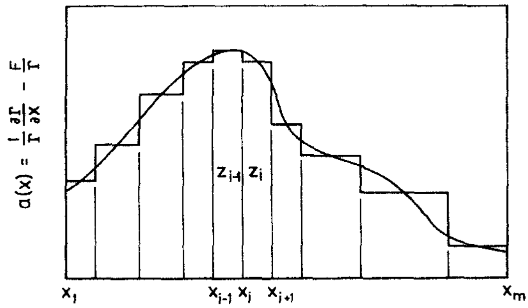

FIG. 1. A schematic profile of $a(x)$ and its step function approximation.

Based on this approximation, the analytical solution of equation (1) in the interval $[x_i, x_{i+1}]$ is

$$\theta(x) = \theta_i + (\theta_{i+1} - \theta_i)\frac{1-\exp(-a_{i+1/2}\xi)}{1-\exp(-a_{i+1/2}\Delta x_i)} \quad (4)$$

where $\xi = x - x_i$ and $0 \leqslant \xi \leqslant \Delta x_i$. For a similar reasoning, the local analytical solution in the interval $[x_{i-1}, x_i]$ is

$$\theta(x) = \theta_{i-1} + (\theta_i - \theta_{i-1})\frac{1-\exp(-a_{i-1/2}\xi^*)}{1-\exp(-a_{i-1/2}\Delta x_{i-1})} \quad (5)$$

where $\xi^* = x - x_{i-1}$ and $0 \leqslant \xi^* \leqslant \Delta x_{i-1}$. From equations (4) and (5), one sees that the solution of equation (1) is expressible with piecewise continuous exponential functions for the entire domain $x_1 \leqslant x \leqslant x_m$. The continuity at the grid point x_i can be verified by substituting $\xi = 0$ and $\xi^* = \Delta x_{i-1}$, respectively, into equations (4) and (5). Next, the derivative $\theta'(x)$ is expected to have a continuous value at $x = x_i$ as long as the coefficient $a(x)$ is continuous there. Upon evaluating the values of $\theta'(x_i)$ from equations (4) and (5) followed by letting them be equal, one arrives at

$$a_W\theta_{i-1} + a_P\theta_i + a_E\theta_{i+1} = 0 \quad (6)$$

or

$$(\theta'' + a\theta')_i = a_W\theta_{i-1} + a_P\theta_i + a_E\theta_{i+1} \quad (7)$$

where

$$a_W = w_f(-Z_{i-1})/(\Delta x_{i-1}\overline{\Delta x_i})$$
$$a_E = w_f(Z_i)/(\Delta x_i\overline{\Delta x_i})$$
$$a_P = -a_W - a_E \quad (8)$$

and

$$w_f(Z) = Z/(1-e^{-Z})$$
$$Z_i = a_{i+1/2}\Delta x_i$$
$$\overline{\Delta x_i} = (\Delta x_{i-1} + \Delta x_i)/2 = (x_{i+1} - x_{i-1})/2. \quad (9)$$

To conserve CPU time, the factor a_W is evaluated by using one of the important features of the weighting function, i.e.

$$w_f(-Z_{i-1}) = w_f(Z_{i-1}) - Z_{i-1} \quad (10)$$

where the value of $w_f(Z_{i-1})$ has been obtained at the previous point.

Equation (6) implies that the value of θ_i depends on the values of θ_{i-1} and θ_{i+1} in the manner

$$\theta_i = \left(\frac{a_W}{a_W + a_E}\right)\theta_{i-1} + \left(\frac{a_E}{a_W + a_E}\right)\theta_{i+1}. \quad (11)$$

The function $w_f(Z)$ thus is called the weighting function because the weighting factors $a_W/(a_W + a_E)$ and $a_E/(a_W + a_E)$ are related directly to the function $w_f(Z)$. For simplicity, the weighting function can also be evaluated from the power-law approximation [4, 5], i.e.

$$w_f(Z) = [0, (1-0.1|Z|)^5] + [0, Z] \quad (12)$$

where $[a, b]$ denotes the greater of a and b. A comparison between the weighting function (see equation (9)) and the power-law (12) can be found in ref. [7].

It should be noted here that in the derivation of the weighting function scheme on a uniform step size Δx, in ref. [7], the values of $(\theta'' + a\theta')_i$ were estimated by the use of a central difference scheme, i.e.

$$(\theta'' + a\theta')_i \approx [(\theta' + a\theta)_{i+1/2} - (\theta' + a\theta)_{i-1/2}]/\Delta x \quad (13)$$

where the value of $(\theta' + a\theta)_{i+1/2}$ is evaluated from the locally analytic solution (4). Obviously, equation (13) is valid only for the cases of a uniform grid. Approximation (13), unfortunately, also disobeys Rule 4 that states the sum of the weighting factors, a_W, a_P and a_E must be zero [4]. Thus ref. [7] employed a modification such that the scheme satisfies this important rule (see the discussion on p. 4 of ref. [7]). In the present new formulation for the weighting function scheme on a variable grid, a continuity in the value of $\theta'(x_i)$ is assumed instead of the use of the central difference scheme (13). As a result, the new formulation automatically satisfies Rule 4 without the need of any further assumption, see equation (8). This verifies that the modification made to the weighting function scheme in ref. [7] is correct.

The spirit of the weighting function scheme is to convert the value of $(\theta'' + a\theta')$ at the point $x = x_i$ to a relationship between the values of θ_{i-1}, θ_i and θ_{i+1} such that a differential equation can be discretized into an algebraic equation, see equations (7)–(9). It is easy to verify that the weighting function scheme will shift automatically to the central difference scheme as the parameter Z approaches zero and to the upwind scheme as $|Z|$ approaches infinity. Other important characteristics of the weighting function and the weighting function scheme can be found in ref. [7].

Application to conservation equations

As mentioned earlier, the weighting function scheme has been applied satisfactorily to a physical equation without a conservative form such as the Orr–Sommerfeld equation [7–11] and the non-similarity boundary layer flow equations [12, 13]. For an energy (or momentum) conservation equation of the form

$$\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\theta}{\partial x}\right) - F\frac{\partial\theta}{\partial x} = S_p\theta + S_c \quad (14)$$

or

$$\frac{\partial^2\theta}{\partial x^2} + \left(\frac{1}{\Gamma}\frac{d\Gamma}{dx} - \frac{F}{\Gamma}\right)\frac{\partial\theta}{\partial x} = \frac{S_p}{\Gamma}\theta + \frac{S_c}{\Gamma} \quad (15)$$

an application of the weighting function scheme, equations (7)–(9), yields

$$a_W\theta_{i-1} + a_P\theta_i + a_E\theta_{i+1} = a_R \quad (16)$$

where the definitions for a_W and a_E are the same as in equations (8) and

$$a_{\mathrm{P}} = -a_{\mathrm{W}} - a_{\mathrm{E}} - S_{\mathrm{p}}/\Gamma_i$$
$$a_{\mathrm{R}} = S_{\mathrm{c}}/\Gamma_i$$
$$Z_i = \left(\frac{1}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}x} - \frac{F}{\Gamma}\right)_{i+1/2} \Delta x_i. \tag{17}$$

In equation (14), Γ and F stand for, respectively, the thermal conductivity and the mass flow rate. The expression $(F - \mathrm{d}\Gamma/\mathrm{d}x)$ appearing in the coefficient of the first derivative term of equation (15) can be treated as an equivalent mass flow rate. The terms on the right-hand side of equation (14) are the source terms.

For the unsteady three-dimensional conservative equation

$$C\frac{\partial\theta}{\partial t} + F_x\frac{\partial\theta}{\partial x} + F_y\frac{\partial\theta}{\partial y} + F_z\frac{\partial\theta}{\partial z} = \frac{\partial}{\partial x}\left(\Gamma\frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Gamma\frac{\partial\theta}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma\frac{\partial\theta}{\partial z}\right) - (S_{\mathrm{p}}\theta + S_{\mathrm{c}}) \tag{18}$$

the discretization equation based on the weighting function scheme, equations (7)–(9), is

$$a_{\mathrm{W}}\theta_{i-1,j,k} + a_{\mathrm{E}}\theta_{i+1,j,k} + a_{\mathrm{S}}\theta_{i,j-1,k} + a_{\mathrm{N}}\theta_{i,j+1,k} + a_{\mathrm{B}}\theta_{i,j,k-1} + a_{\mathrm{T}}\theta_{i,j,k+1} + a_{\mathrm{P}}\theta_{i,j,k} = a_{\mathrm{R}}$$
$$a_{\mathrm{W}} = w_{\mathrm{f}}(-Z_{x,i-1})/(\Delta x_{i-1}\overline{\Delta x_i})$$
$$a_{\mathrm{E}} = w_{\mathrm{f}}(Z_{x,i})/(\Delta x_i\overline{\Delta x_i})$$
$$a_{\mathrm{S}} = w_{\mathrm{f}}(-Z_{y,j-1})/(\Delta y_{j-1}\overline{\Delta y_j})$$
$$a_{\mathrm{N}} = w_{\mathrm{f}}(Z_{y,j})/(\Delta y_j\overline{\Delta y_j})$$
$$a_{\mathrm{B}} = w_{\mathrm{f}}(-Z_{z,k-1})/(\Delta z_{k-1}\overline{\Delta z_k})$$
$$a_{\mathrm{T}} = w_{\mathrm{f}}(Z_{z,k})/(\Delta z_k\overline{\Delta z_k})$$
$$a_{\mathrm{P}} = -a_{\mathrm{W}} - a_{\mathrm{E}} - a_{\mathrm{S}} - a_{\mathrm{N}} - a_{\mathrm{B}} - a_{\mathrm{T}} - (S_{\mathrm{p}} + C/p\Delta t)_{i,j,k}/\Gamma_{i,j,k}$$
$$a_{\mathrm{R}} = (S_{\mathrm{c}} - C\bar{\theta}_0/p\Delta t)_{i,j,k}/\Gamma_{i,j,k}$$
$$Z_{x,i} = \left(\frac{1}{\Gamma}\frac{\partial\Gamma}{\partial x} - \frac{F_x}{\Gamma}\right)_{i+1/2} \Delta x_i, \text{ etc.} \tag{19}$$

where subscripts W, E, S, N, B, T, P and R denote, respectively, west, east, south, north, bottom, top, the present point and the right-hand side. Subscripts i, j, k stand for a quantity at the location (x_i, y_j, z_k).

In the formulation of equations (19), the unsteady term $\partial\theta/\partial t$ is discretized by using the p-method proposed in ref. [15], i.e.

$$\partial\theta/\partial t = (\theta - \bar{\theta}_0)/p\Delta t$$
$$\bar{\theta}_0 = \theta_0 + \theta'_0(1-p)\Delta t \tag{20}$$

where $\theta' = \partial\theta/\partial t$ and subscript '0' denotes quantities at the previous time $t - \Delta t$. Formulation (20) reduces to the backward difference scheme in the time coordinate if $p = 1$ is assigned.

WEIGHTING FUNCTION SCHEME VS EXPONENTIAL SCHEME

The weighting function scheme presented in equations (7)–(9) looks similar to Patankar's exponential scheme [4] at first glance, because both of them are in terms of an exponential function. However, they are very different in some ways. For convenience of comparison, Patankar's exponential scheme for equation (14) without the source terms is expressed with the notations employed in the present investigation as

$$a_{\mathrm{W}} = \frac{F_{\mathrm{w}}\exp(F_{\mathrm{w}}\Delta x_{i-1}/\Gamma_{\mathrm{w}})}{\exp(F_{\mathrm{w}}\Delta x_{i-1}/\Gamma_{\mathrm{w}}) - 1}$$
$$a_{\mathrm{E}} = \frac{F_{\mathrm{e}}}{\exp(F_{\mathrm{e}}\Delta x_i/\Gamma_{\mathrm{e}}) - 1}$$
$$a_{\mathrm{P}} = -a_{\mathrm{W}} - a_{\mathrm{E}} + (F_{\mathrm{e}} - F_{\mathrm{w}}) \tag{21}$$

where the subscripts w and e denote quantities at $x_{i-1/2}$ and $x_{i+1/2}$, respectively. However, the thermal conductivity is defined by the harmonic mean scheme [6], i.e.

$$\Gamma_{\mathrm{w}} = 2\Gamma_i\Gamma_{i-1}/(\Gamma_i + \Gamma_{i-1})$$
$$\Gamma_{\mathrm{e}} = 2\Gamma_i\Gamma_{i+1}/(\Gamma_i + \Gamma_{i+1}). \tag{22}$$

With this, the major differences between the weighting function scheme and Patankar's exponential scheme are discussed as follows.

Satisfaction on Rule 4

As mentioned earlier, the present weighting function scheme always obeys Rule 4 stated in Chapter 3 of ref. [4]. Patankar's exponential scheme, however, disobeys the rule unless $F_{\mathrm{e}} = F_{\mathrm{w}}$ as can be seen from equation (21). It appears that in Patankar's exponential scheme, the heat flux is not necessarily continuous at point x_i, even though the overall heat transfer satisfies the conservation law between the two control surfaces of the control volume. Fortunately, in continuum fluid flows, the mass flow rate F always satisfies the continuity equation such that the term $(F_{\mathrm{e}} - F_{\mathrm{w}})$ appearing in equation (21) can be removed. Therefore, it must be emphasized that the application of Patankar's exponential scheme should be restricted to conservation problems. The new weighting function scheme formulation proposed in the present paper does not have such a limitation. In addition, it can be applied to complex variables as well. In the present formulation of the weighting function scheme, the value of $\theta'(x)$ is assumed to be continuous at point x_i. Such a procedure is equivalent to employing a control volume of zero size enclosing the point x_i. This might account for the fact that the present formulation for the weighting function scheme always satisfies Rule 4.

Variable thermal conductivity

Physically speaking, the effect of a variable thermal conductivity on the temperature distribution is similar to that of a fluid flow. This can be seen by considering a one-dimensional heat conduction problem in the domain $0 \leqslant x \leqslant 1$ under the boundary conditions $\theta(0) = 1$ and $\theta(1) = 0$. For such a problem, the exact temperature at $x = 0.5$ is $\theta(0.5) = 0.5$ if the thermal conductivity is uniform in the entire domain. However, the value of $\theta(0.5)$ will be less than 0.5 when the thermal conductivity assumes an increasing value in the positive x-direction, i.e. $\mathrm{d}\Gamma/\mathrm{d}x > 0$. For the limiting case of $\mathrm{d}\Gamma/\mathrm{d}x = \infty$, the value of $\theta(0.5)$ will even become zero due to a 'perfect' conductivity in the region $0.5 \leqslant x \leqslant 1$ such that the boundary condition at $x = 1$ dominates the value of $\theta(0.5)$. This situation is equivalent to the upwind effect in convective heat transfer problems. Such a physical phenomenon can also be verified mathematically from a comparison between equations (14) and (15).

It is very important to note here that the weighting function scheme cannot solve equation (14) directly, because equation (14) does not possess the form of equation (1). Thus, before applying the weighting function scheme, a conservation equation must be transformed into a non-conservative form as shown in equation (15). In contrast, Patankar's exponential scheme does not apply to the non-conservative form (15), because the equivalent mass flow rate $(F - \partial\Gamma/\partial x)$ does not necessarily satisfy the 'pseudo-continuity' equation unless the thermal conductivity is a harmonic function ($\nabla^2\Gamma = 0$). Therefore, the weighting function scheme must consider the net effect of the variable thermal conductivity and the fluid flow, i.e. $(\mathrm{d}\Gamma/\mathrm{d}x - F)/\Gamma$. In the use of Patankar's exponential scheme, however, the effects of fluid flow and variable thermal conductivity should be treated separately. The exponential function is employed to treat the effect of fluid flow, whereas the harmonic mean formulation [6] is used to account for the effect of variable thermal conductivity.

As a final note in this section, it is mentioned that the spirit of the weighting function scheme is entirely different from that of Patankar's exponential scheme, although both of them assume exponential variation in the solution. The weighting function scheme converts the sum of the first and the second derivative terms of an unknown function θ at the point $x = x_i$, i.e. $(\theta'' + a\theta')_i$, into a relationship among the θ-values at that particular point x_i and its two nearest neighbours. The weighting function then is employed to determine their weightings with the parameter Z. From equation (9) and Fig. 1, one sees that the parameter Z_i approximates the area under the curve $a(x)$ in the interval $[x_i, x_{i+1}]$. Therefore, Z_i stands for the *grid Peclet number* in that interval. In Patankar's exponential scheme, the conservation law for a control volume of finite size is emphasized. However, an artificial thermal conductivity is employed instead of the real one at the control surface. Such a treatment does not have a clear physical significance.

PERFORMANCE OF THE WEIGHTING FUNCTION SCHEME

The weighting function scheme has been developed in the present investigation for conservation equations, see equations (18) and (19). For the case of uniform thermal conductivity, the weighting function scheme becomes Patankar's exponential scheme [4]. Hence, the weighting function scheme is very easy to use along with the pressure correction equation in an algorithm such as SIMPLE and SIMPLER [4], etc. As discussed in the previous section, for a conservation problem, the major difference between the weighting function scheme and Patankar's exponential scheme is the treatment for a variable thermal conductivity. Definitely, the use of harmonic mean formulation in Patankar's exponential scheme will result in a different accuracy for the solution especially when the thermal conductivity has a strong variation. To compare the performance of the weighting function scheme with that of Patankar's exponential scheme, several heat conduction problems are illustrated with an emphasis on the effect of variable thermal conductivity. Note that Patankar's exponential scheme reduces to the harmonic mean scheme [6] for heat conduction problems. Hence, the harmonic mean scheme will be used instead of Patankar's exponential scheme in the following examples.

Example 1. *One-dimensional heat conduction with* $\Gamma = e^{ax}$

The governing equation for one-dimensional heat conduction with a thermal conductivity of $\Gamma = e^{ax}$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{ax}\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) = 0 \tag{23}$$

which has the exact solution

$$\theta(x) = \frac{1 - e^{a(1-x)}}{1 - e^{a}} \tag{24}$$

when the boundary conditions are $\theta(0) = 1$ and $\theta(1) = 0$. To compare the performance of available numerical methods, this heat conduction problem was solved by using the weighting function scheme, the weighting function scheme with power-law approximation (equation (12)), the harmonic mean scheme [6], the central difference scheme, the box method [16] and a fourth-order Runge–Kutta method. All of the methods except for the harmonic mean scheme solve the equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + a\frac{\mathrm{d}\theta}{\mathrm{d}x} = 0 \tag{25}$$

instead of equation (23), because they are all non-conservative schemes. The results based on $\Delta x = 0.1$ and $m = 11$ points are listed in Table 1 for $a = 10$, 20 and 50.

From Table 1, one sees that the weighting function

Table 1. Comparisons of the $\theta(x)$ results among various schemes for Example 1 ($\Delta x = 0.1$)

a	x	WFS†	WFS—PLA‡	CDS§	Box‖	RKM¶
	0.0	1.0000	1.0000	1.0000	1.0000	1.0000
	0.1	0.3679	0.3712	0.3333	0.3333	0.3750
	0.2	0.1353	0.1378	0.1111	0.1111	0.1406
	0.3	0.0497	0.0511	0.0370	0.0370	0.0527
	0.4	0.0183	0.0190	0.0123	0.0123	0.0197
10	0.5	0.0067	0.0070	0.0041	0.0041	0.0074
	0.6	0.0024	0.0026	0.0014	0.0014	0.0027
	0.7	0.0009	0.0009	0.0004	0.0004	0.0010
	0.8	0.0003	0.0003	0.0001	0.0001	0.0003
	0.9	0.0001	0.0001	0.0000	0.0000	0.0001
	1.0	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0	1.0000	1.0000	1.0000	1.0000	1.0000
	0.1	0.1353	0.1408	0.0000	0.0000	0.3333
	0.2	0.1832	0.1982	0.0000	0.0000	0.1111
	0.3	0.0025	0.0028	0.0000	0.0000	0.0370
	0.4	0.0003	0.0004	0.0000	0.0000	0.0123
20	0.5	0.0000	0.0001	0.0000	0.0000	0.0041
	0.6	0.0000	0.0000	0.0000	0.0000	0.0014
	0.7	0.0000	0.0000	0.0000	0.0000	0.0005
	0.8	0.0000	0.0000	0.0000	0.0000	0.0002
	0.9	0.0000	0.0000	0.0000	0.0000	0.0001
	1.0	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0	1.0000	1.0000	1.0000	1.0000	1.0000(00)††
	0.1	0.0067	0.0062	−0.4289	−0.4289	0.1371(02)
	0.2	0.0000	0.0000	0.1835	0.1835	0.1879(03)
	0.3	0.0000	0.0000	−0.0789	−0.0789	0.2576(04)
	0.4	0.0000	0.0000	0.0335	0.0335	0.3531(05)
50	0.5	0.0000	0.0000	−0.0147	−0.0147	0.4841(06)
	0.6	0.0000	0.0000	0.0060	0.0060	0.6636(07)
	0.7	0.0000	0.0000	−0.0029	−0.0029	0.9097(08)
	0.8	0.0000	0.0000	0.0009	0.0009	0.1247(10)
	0.9	0.0000	0.0000	−0.0007	−0.0007	0.1710(11)
	1.0	0.0000	0.0000	0.0000	0.0000	0.2343(12)

† Weighting function scheme, also, the harmonic mean scheme and the exact solution.
‡ Weighting function scheme with power-law approximation.
§ Central difference scheme.
‖ Box method.
¶ Fourth-order Runge–Kutta method.
†† $\alpha(\beta) = \alpha \times 10^{\beta}$.

scheme always provides the exact solution. This is because the only approximation (equation (3)) made for the weighting function scheme just becomes exact if the coefficient a in equation (25) is a constant. However, an absolute error of 0.0150 arises at $a = 20$ and $x = 0.2$ when the power-law (12) is used to approximate the weighting function. The central difference scheme yields $a_W = 1 - a\Delta x/2$, $a_E = 1 + a\Delta x/2$ and $a_P = -2$ for equation (25). From Table 1, the central difference scheme is seen to produce a reasonable result for $a = 10$. Unfortunately, for $a = 20$ the result becomes $\theta(x) = 0$ due to $a_W = 0$. An oscillating solution can even be observed when a is increased to 50 such that the value of a_W becomes negative. The same results are also predicted by the box method [16]. In convective heat transfer problems, the Reynolds number (or Peclet number) that usually appears in the coefficient of the first derivative term has a magnitude of $O(10^3)$ in general. Thus, an unrealistic solution might arise when the central difference scheme or the box method is employed. In the use of the Runge–Kutta method, the boundary condition $\theta'(0) = -a/(1-e^a)$ is given instead of $\theta(1) = 0$. The latter thus is not necessarily satisfied owing to numerical errors. The fourth-order Runge–Kutta method shows a divergent solution at $a = 50$.

It is interesting to note that the harmonic mean scheme [6] also produces the exact solution for the present example. However, the harmonic mean scheme could produce a solution with a significant error for problems having a variable thermal conductivity without an exponential form. This point will be discussed in Example 2.

Example 2. *One-dimensional heat conduction with* $\Gamma = (1+x^2)^a$

Consider the one-dimensional heat conduction problem

Table 2. Comparisons of the $\theta(x)$ results among the weighting function scheme (WFS), the central difference scheme (CDS) and the harmonic mean scheme (HMS) for Example 2 ($\Delta x = 0.1$)

a	x	Exact	WFS	CDS	HMS
100	0.0	1.0000	1.0000	1.0000	1.0000
100	0.1	0.1599	0.1629	0.1537	0.2302
100	0.2	0.0053	0.0055	−0.0183	0.0113
100	0.3	0.0000	0.0000	0.0056	0.0001
100	0.4	0.0000	0.0000	−0.0022	0.0000
100	0.5	0.0000	0.0000	0.0013	0.0000
100	0.6	0.0000	0.0000	−0.0006	0.0000
100	0.7	0.0000	0.0000	0.0005	0.0000
100	0.8	0.0000	0.0000	−0.0001	0.0000
100	0.9	0.0000	0.0000	0.0003	0.0000
100	1.0	0.0000	0.0000	0.0000	0.0000
1	0.1	0.8731	0.8731	0.8732	0.8732
5	0.1	0.7675	0.7677	0.7681	0.7693
10	0.1	0.6677	0.6681	0.6689	0.6729
20	0.1	0.5359	0.5365	0.5384	0.5496
50	0.1	0.3222	0.3235	0.3257	0.3633
100	0.1	0.1599	0.1629	0.1537	0.2302

$$\frac{d}{dx}\left[(1+x^2)^a \frac{d\theta}{dx}\right] = 0, \quad \theta(0) = 1 \quad \text{and} \quad \theta(1) = 0 \tag{26}$$

which has a thermal conductivity of the form $\Gamma = (1+x^2)^a$. A strong variation in Γ thus can be obtained by assigning a large value to a. The harmonic mean scheme [6] is used to solve equations (26), whereas the weighting function scheme and the central difference scheme are employed to solve a non-conservative form of this same problem, i.e.

$$\frac{d^2\theta}{dx^2} + \frac{2ax}{1+x^2}\frac{d\theta}{dx} = 0, \quad \theta(0) = 1 \quad \text{and} \quad \theta(1) = 0. \tag{27}$$

The results based on $\Delta x = 0.1$ and $m = 11$ points are presented in Table 2 for $a = 1$, 5, 10, 20, 50 and 100. For the purpose of comparisons, the results evaluated from the exact solution

$$\theta(x) = 1 - f_a(x)/f_a(1)$$

$$f_1(x) = \tan^{-1} x$$

$$f_{a+1}(x) = \frac{x}{2a(1+x^2)^a} + \frac{2a-1}{2a} f_a(x) \tag{28}$$

are also provided in Table 2. From Table 2, it is seen that the weighting function scheme produces accurate results even when Γ has a very strong variation at $a = 100$. The central difference scheme, again, results in an oscillating solution at $a = 100$. The harmonic mean scheme [6] overpredicts the result by a great amount when Γ has a strong variation. For instance, for $a = 100$ the harmonic mean scheme gives $\theta(0.1) = 0.2302$ as compared to 0.1599 from the exact solution and 0.1629 from the weighting function scheme. Such an error will be examined in Example 3 for the two-dimensional case.

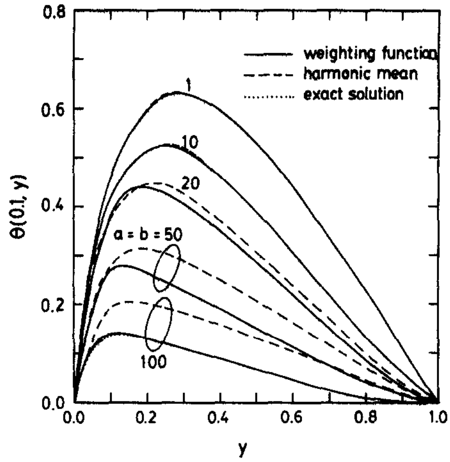

FIG. 2. Comparisons of $\theta(0.1, y)$ results between the weighting function scheme and the harmonic mean scheme for Example 3 ($\Delta x = \Delta y = 0.1$).

Example 3. *Two-dimensional heat conduction with* $\Gamma = (1+x^2)^a(1+y^2)^b$

The performance of the weighting function scheme is to compare with that of the harmonic mean scheme [6] through the two-dimensional heat conduction problem

$$\frac{\partial}{\partial x}\left[(1+x^2)^a(1+y^2)^b \frac{\partial\theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[(1+x^2)^a(1+y^2)^b \frac{\partial\theta}{\partial y}\right] = 0$$

$$\theta(0,y) = \cos(\pi y/2), \quad \theta(1,y) = 0$$

$$\theta(x,0) = 0, \quad \theta(x,1) = 0. \tag{29}$$

The results of $\theta(0.1, y)$ based on $\Delta x = \Delta y = 0.1$ from both schemes are shown in Fig. 2 for $a = b = 1$, 10, 20, 50 and 100. The 'exact' solution for the case of $a = b = 100$ is also plotted in Fig. 2 for comparisons. Note that the 'exact' solution was obtained by reducing the step size until four-place accuracy was achieved. The 'exact' solutions for $\theta(0.1, 0.1)$, $\theta(0.1, 0.2)$ and $\theta(0.1, 0.3)$ are, respectively, 0.1345, 0.1255 and 0.1027 based on the weighting function scheme and 0.1368, 0.1319 and 0.1113 based on the harmonic mean scheme. The agreement between the two 'exact' solutions is so good that their difference is difficult to show in Fig. 2. Again, one sees from Fig. 2 that the harmonic mean scheme overpredicts the result by a great amount for the case of $a = b = 100$. The weighting function scheme, however, still produces accurate results even when Γ has such a strong variation. In the weighting function scheme, the thermal conductivity Γ is assumed to have an exponential variation between two grid points, i.e. $(1/\Gamma)(\partial\Gamma/\partial x) =$ constant. In contrast, the harmonic mean scheme approximates the thermal conductivity with a step function. This may account for the fact that the weighting function scheme has a superior performance in Examples 2 and 3.

In the above examples, one might wish to apply Patankar's exponential scheme to the non-conservative form (equations (27)) to avoid the use of the harmonic mean scheme. Unfortunately, Patankar's exponential scheme solves the non-conservative form only when the thermal conductivity happens to be a harmonic function as mentioned earlier. In many physical problems such as a highly turbulent flow, the eddy viscosity (or eddy diffusivity in turbulent heat transfer) Γ possesses a sharp variation in the wall region due to the existence of a viscous sublayer adjacent to the solid surface. Under such a situation, the weighting function scheme might produce more accurate results than Patankar's exponential scheme does for a given grid size. The power-law [4, 5], a well-known scheme, is an approximation to the exponential scheme. Therefore, its accuracy is expected to be no better than that of the exponential scheme.

CONCLUSION

The weighting function scheme proposed previously has shown excellent performance for physical problems without conservation equations such as the wave instability and the non-similarity boundary layer flow equations. This numerical scheme is derived on a mathematical basis rather than on a control volume approach such that it can be applied on almost any physical problem with and without conservation equations. However, in the previous formulation of the weighting function scheme, a modification must be made to force the scheme to obey a particular numerical rule. In the present investigation, the weighting function scheme is reformulated based on a different point of view such that the new formulation always satisfies the numerical rule without the need of any further assumption. In addition, the new formulation allows the weighting function scheme to apply on grid systems of variable step size. When applied to the conservation equations, the weighting function scheme becomes Patankar's exponential scheme if the thermal conductivity is uniform in the entire domain. For cases of variable thermal conductivity, the weighting function scheme approximates the thermal conductivity with an exponential function, whereas Patankar's exponential scheme assumes the thermal conductivity to be a step function. Hence, the weighting function scheme has a performance superior to that of Patankar's exponential scheme as can be observed from the examples illustrated in the present study. In many physical problems such as turbulent heat transfer, the eddy diffusivity Γ has a strong variation in the wall region due to the existence of a viscous sublayer adjacent to the solid surface. Under such a situation, the weighting function scheme is expected to produce results more accurate than that based on Patankar's exponential scheme.

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LE SCHEMA AVEC FONCTION DE PONDERATION ET SON APPLICATION AUX EQUATIONS MULTIDIMENSIONNELLES DE CONSERVATION

Résumé—Le schéma avec fonction de pondération déjà proposé a montré beaucoup de succès dans la résolution des problèmes qui n'ont pas une forme conservative comme ceux d'instabilité et les équations d'écoulement à couche limite non affine. Néanmoins la grille était limitée à un pas uniforme et il faut apporter une modification pour obéir à une importante règle numérique. Une nouvelle formulation est proposée pour la fonction de pondération de façon à supprimer la contrainte d'un pas uniforme. Elle garantit l'application de la règle numérique sans nécessiter une hypothèse quelconque. Le schéma avec fonction de pondération devient alors le schéma exponentiel de Patankar dans le cas d'une conductivité thermique uniforme. Dans le cas d'une conductivité variable, ce schéma à fonction de pondération est supérieur à celui de Patankar et il semble pouvoir être performant pour traiter un écoulement turbulent près d'une paroi où la viscosité turbulente varie fortement du fait de la sous-couche visqueuse adjacente à la surface solide.

EIN SYSTEM VON GEWICHTUNGSFUNKTIONEN UND SEINE ANWENDUNG AUF MEHRDIMENSIONALE ERHALTUNGSGLEICHUNGEN

Zusammenfassung—Ein früher vorgeschlagenes Verfahren mit Gewichtungs-Funktionen hat sich beim Lösen physikalischer Probleme nichtkonservativer Form bewährt, wie z. B. bei Problemen der Welleninstabilität und der nichtähnlichen Gleichungen für Grenzschichtströmungen. Das bisher verwendete Verfahren mit Gewichtungs-Funktionen ist auf einheitliche Gitterabstände beschränkt; es muß eine Anpassung vorgenommen werden, damit ein wichtiges numerisches Kriterium erfüllt wird. In der vorliegenden Arbeit wird eine neue Form vorgeschlagen, welche die Beschränkung auf ein gleichmäßiges Gitter aufhebt. Desweiteren wird das numerische Kriterium ohne zusätzliche Anpassung erfüllt. Bei Anwendung auf die Erhaltungssätze erhält man aus dem System der Gewichtungs-Funktionen Patankar's Exponentialsystem für den Fall einheitlicher Wärmeleitfähigkeit. Im Falle variabler Wärmeleitfähigkeit ist das hier vorgestellte Verfahren dem Patankar'schen Exponentialsystem überlegen. Es wird daher erwartet, daß das neue Verfahren gut bei der Beschreibung turbulenter wandnaher Strömungen angewandt werden kann, bei der sich die turbulente Zähigkeit infolge einer viskosen Unterschicht in Nähe der festen Oberfläche abrupt ändert.

СХЕМА ВЕСОВЫХ ФУНКЦИЙ И ЕЕ ПРИМЕНЕНИЕ К МНОГОМЕРНЫМ УРАВНЕНИЯМ СОХРАНЕНИЯ

Аннотация—Предложенная ранее схема весовых функций успешно применялась при решении физических задач в неконсервативныной форме типа расчета неавтомодельных уравнений о течении в пограничном слое и задач о неустойчивости волн. Однако в предыдущей формулировке сетка должна была быть равномерной, и, чтобы схема удовлетворяла важному числовому правилу, необходима была ее модификация. В данном исследовании предложен новый вариант схема весовых функций, при котором можно отказаться от ограничения равномерности сетки. Кроме того, новая формулировка обеспечивает удовлетворение схемы весовых функций числовому правилу без дополнительных предположений. При реализации схемы весовых функций для уравнений сохранения она переходит в экспоненциальную схему Патанкара при условии постоянного коэффициента теплопроводности. В случае переменного коэффициента теплопроводности схема весовых функций предпочтительнее схемы Патанкара. Таким образом, следует ожидать, что схема весовых функций более эффективна при исследовании пристенного турбулентного течения, когда вихревая вязкость резко изменяется из-за наличия вязкого подслоя вблизи твердой стенки.